3.5: Fundamental Theorem of Algebra

Fundamental Theorem of Algebra: if the degree of a polynomial is n, then there are n (not necessarily distinct) zeros.

Recall:

1. Complex roots come in pairs.
2. $i = \sqrt{-1} \quad and \quad i^2 = -1$

Factor down to linear factors.

$$4x^3 - 12x^2 + 9x - 27$$

$$4x^2(x-3) + 9(x-3)$$

$$(4x^2+9)(x-3)$$

imaginary ✓

$$(2x-3i)(2x+3i)(x-3)$$

Find all zeros and factor down to linear factors.

$$x^4 - x^3 + x^2 + 9x - 10$$

$$\begin{array}{r|rrrrr}
1 & 1 & -1 & 1 & 9 & -10 \\
 & \downarrow & 1 & 0 & 1 & 10 \\
\hline
-2 & 1 & 0 & 1 & 10 & 0 \\
 & \downarrow & -2 & 4 & -10 & \\
\hline
 & 1 & -2 & 5 & 0 &
\end{array}$$

Zeros

$\{1, -2, 1 \pm 2i$

$$(x-1)(x+2)(x^2-2x+5)$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Linear:

$$\boxed{(x-1)(x+2)(x-1-2i)(x-1+2i)}\ \checkmark$$

Use the given zero to find the remaining zeros of each function.

26) $3x^4 + 5x^3 + 25x^2 + 45x - 18$

$i^2 = -1$ $3i$

$\boxed{\text{Zeros: } 3i, -3i, 1/3, -2}$

$$\begin{array}{r|rrrrr}
3i & 3 & 5 & 25 & 45 & -18 \\
 & \downarrow & 9i & 15i + \cancel{27i^2}\ -27 & -6i - 45 & 18 \\
\hline
-3i & 3 & 5+9i & -2+15i & -6i & \boxed{0} \\
 & \downarrow & -9i & -15i & 6i & \\
\hline
 & 3 & 5 & -2 & \boxed{0} &
\end{array}$$

$3x^2 + 5x - 2$

$(3x - 1)(x + 2)$

$x = 1/3$ $\quad -2$

Use the given zero to find the remaining zeros of each function.

$$2x^3 - 13x^2 + 30x - 25; \quad 2 + i$$

Zeros: $2+i$, $2-i$, $5/2$

$(x - 2 - i)(x - 2 + i)$

$x \rightarrow$ $x^2 - 2x + \cancel{xi} + 4 - \cancel{2i}$

$-2 \rightarrow$ $-2x$

$-i \rightarrow$ $-\cancel{xi} + \cancel{2i} - i^2$

$x^2 - 4x + 4 + 1$

$2x - 5 = 0$

$$\begin{array}{r} 2x - 5 \\ x^2 - 4x + 5 \;\big)\; \cancel{2x^3} - 13x^2 + 30x - 25 \\ -(\cancel{2x^3} - 8x^2 + 10x) \\ \hline -5x^2 + 20x - 25 \\ -5x^2 + 20x - 25 \\ \hline 0 \end{array}$$